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THE RATIONAL FUNCTIONS OF THE CUBIC.

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The more common solutions of the cubic are those where a supposition is made to exist between auxiliary variables introduced in order to effect its reduction. It is the object of this paper to give the solution without this expedient, depending simply upon the relations the roots sustain to the coefficients and for another reason, much more important, to give the solution of the cubic in precisely the same manner as the quadratic is solved. In order that the work may be progressive and connected, the quadratic

$$x^2 + Ax + B = 0\,,$$

whose roots are $x = a$, $x = b$, and whose equation is

$$x^2 - (a + b)x + ab = 0\,,$$

will be solved by the method that, if not the briefest, I still maintain is the simplest and most natural of any method known, and will be introductory to the method given for the cubic.

Equating, we have

$$a + b = -A\,,$$

$$ab = B\,.$$

Squaring the first and subtracting four times the second from it, we obtain

$$a^2 - 2ab + b^2 = A^2 - 4B\,,$$

whence, extracting the square root

$$a - b = \pm\sqrt{A^2 - 4B}\,.$$

Combining this equation with the one above by addition and subtraction, we have

$$a = -\frac{A}{2} + \sqrt{\frac{1}{4}A^2 - 41}\,,$$

$$b = -\frac{A}{2} - \sqrt{\frac{1}{4}A^2 - B}\,;$$

and these being the values of a and b, are the values of x by hypothesis, and so are the roots of the quadratic equation.

Assume the cubic in the form

$$x^3 + Ax + B = 0\,.$$

Since the sum of the roots is zero, one is the negative sum of the other two, and we have the roots expressed as $x = a$, $x = b$, $x = -(a + b)$. Whence

$$x - a = 0\,, \quad x - b = 0\,, \quad x + a + b = 0\,,$$

whose cubic equation is

$$x^3 - (a^2 + ab + b^2)\,x + ab\,(a + b) = 0\,.$$

Whence, equating with the equation above,

$$-\frac{1}{3}(a^2 + ab + b^2) = \frac{A}{3}\,, \tag{1}$$

$$\frac{1}{2}(a + b)\,ab = \frac{B}{2}\,. \tag{2}$$

Cubing the first,

$$-\frac{1}{27}a^6 - \frac{1}{9}a^5b - \frac{2}{9}a^4b^2 - \frac{7}{27}a^3b^3 - \frac{2}{9}a^2b^4 - \frac{1}{9}ab^5 - \frac{1}{27}b^6 = \frac{A^3}{27}\,.$$

Squaring the second,

$$\frac{1}{4}a^4b^2 + \frac{1}{2}a^3b^3 + \frac{1}{4}a^2b^4 = \frac{B^2}{4}\,.$$

Adding these equations we obtain

$$-\frac{1}{27}a^6 - \frac{1}{9}a^5b + \frac{1}{36}a^4b^2 + \frac{13}{54}a^3b^3 + \frac{1}{36}a^2b^4 - \frac{1}{9}ab^5 - \frac{1}{27}b^6 = \frac{A^3}{27} + \frac{B^2}{4}\,.$$

Factoring from the first member $-\frac{3}{64}$,

$$-\frac{3}{64}\left[\frac{64}{81}a^6 + \frac{64}{27}a^5b - \frac{64}{108}a^4b^2 - \frac{832}{162}a^3b^3 - \frac{64}{108}a^2b^4 + \frac{64}{27}ab^5 + \frac{64}{81}b^6\right] = \frac{A^3}{27} + \frac{B^2}{4}\,.$$

Extracting the square root of this perfect square,

$$\frac{1}{8}\sqrt{-3}\left[\frac{8}{9}a^3 + \frac{4}{3}a^2b - \frac{4}{3}ab^2 - \frac{8}{9}b^3\right] = \sqrt{\frac{A^3}{27} + \frac{B^2}{4}}\,. \tag{3}$$

Adding $\frac{1}{8}(a^3 + b^3)$ to equation (2), and also subtracting the same, we have

$$-\frac{1}{8}(a^3 + 3a^2b + 3ab^2 + b^3 - a^3 + a^2b + ab^2 - b^3) = -\frac{B}{2}\,. \tag{4}$$

Adding equations (4) and (3) we have

$$-\frac{1}{8}\Big[a^3+3a^2b+3ab^2+b^3-(a^3+a^2b-ab^2-b^3)\sqrt{-3}-a^3+a^2b+ab^2-b^3$$
$$+\left\{\frac{1}{9}a^3-\frac{1}{3}a^2b+\frac{1}{3}ab^2-\frac{1}{9}b^3\right]\sqrt{-3}\Big]=-\frac{B}{2}+\sqrt{\frac{A^3}{27}+\frac{B^2}{4}}.$$

Extracting the cube root of this perfect cube,

$$-\frac{a+b}{2}+\frac{a-b}{2\sqrt{-3}}=\sqrt[3]{-\frac{B}{2}+\sqrt{\frac{A^3}{27}+\frac{B^2}{4}}}. \qquad (5)$$

Subtracting (4) from (3),

$$-\frac{1}{8}\Big[a^3+3a^2b+3ab^2+b^3+(a^3+a^2b-ab^2-ab^3)\sqrt{-3}-a^3+a^2b+ab^2-b^3$$
$$-\left\{\frac{1}{9}a^3-\frac{1}{3}a^2b+\frac{1}{3}ab^2-\frac{1}{9}b^3\right]\sqrt{-3}=-\frac{B}{2}-\sqrt{\frac{A^3}{27}+\frac{B^2}{4}}.$$

Extracting the cube root of this perfect cube,

$$-\frac{a+b}{2}-\frac{a-b}{2\sqrt{-3}}=\sqrt[3]{-\frac{B}{2}-\sqrt{\frac{A^3}{27}+\frac{B^2}{4}}}. \qquad (6)$$

Adding (5) and (6),

$$-(a+b)=\sqrt[3]{-\frac{B}{2}+\sqrt{\frac{A^3}{27}+\frac{B^2}{4}}}+\sqrt[3]{-\frac{B}{2}-\sqrt{\frac{A^3}{27}+\frac{B^2}{4}}}. \qquad (7)$$

Subtracting (6) from (5),

$$a-b=\sqrt{-3}\sqrt[3]{-\frac{B}{2}+\sqrt{\frac{A^3}{27}+\frac{B^2}{4}}}-\sqrt{-3}\sqrt[3]{-\frac{B}{2}-\sqrt{\frac{A^3}{27}+\frac{B^2}{4}}}.$$

Whence, by comparison of these two equations,

$$a=\left\{-\frac{1}{2}+\frac{1}{2}\sqrt{-3}\right]\sqrt[3]{-\frac{B}{2}+\sqrt{\frac{A^3}{27}+\frac{B^2}{4}}}$$
$$+\left\{-\frac{1}{2}-\frac{1}{2}\sqrt{-3}\right]\sqrt[3]{-\frac{B}{2}-\sqrt{\frac{A^3}{27}+\frac{B^2}{4}}}, \qquad (8)$$

$$b=\left\{-\frac{1}{2}-\frac{1}{2}\sqrt{-3}\right]\sqrt[3]{-\frac{B}{2}+\sqrt{\frac{A^3}{27}+\frac{B^2}{4}}}$$
$$+\left\{-\frac{1}{2}+\frac{1}{2}\sqrt{-3}\right]\sqrt[3]{-\frac{B}{2}-\sqrt{\frac{A^3}{27}+\frac{B^2}{4}}}. \qquad (9)$$

Thus we arrive at the values of $-(a+b)$, a, and b, equations (7), (8) and (9), and these being the three values of x are the three roots of the cubic equation. The solution, it will be observed, is the analogue of the solution of the quadratic given above. The quartic is readily reduced in the same manner.

Now several important properties of the formula for the cubic may be pointed out, which, so far as I have observed, have hitherto not been given.

The three elements of the formula are obviously

$$\sqrt{\frac{A^3}{27}+\frac{B^2}{4}},\quad \sqrt[3]{-\frac{B}{2}+\sqrt{\frac{A^3}{27}+\frac{B^2}{4}}},\quad \text{and}\quad \sqrt[3]{-\frac{B}{2}-\sqrt{\frac{A^3}{27}+\frac{B^2}{4}}}.$$

It may now be asked what are the rational functions of the roots of the cubic that correspond to each one of these elements. Those who have perused Wantzel's Abstract of Abel's Argument (Serret's Algébre Superieure, Vol. II, p. 393) are aware that he assumed that such rational functions existed theoretically. For the first of the above forms we have, at once, from equation (3),

$$a^3+\frac{3}{2}a^2b-\frac{3}{2}ab^2-b^3=\sqrt{-27\left[\frac{A^3}{27}+\frac{B^2}{4}\right]}. \tag{10}$$

Factoring the first member,

$$(a-b)\left[a+\frac{1}{2}b\right](a+2b)=\sqrt{-A^3-\frac{27}{4}B^2}.$$

Or the rational functions of the roots of the cubic corresponding to the *square root radical* of the formula in the product of three linear factors in an arithmetical progression of which the first is the difference of the independent roots, and the common difference in $\frac{3}{2}b$, one of those roots.

It is here useful to observe that in the quadratic the radical corresponds simply to the difference of the roots, which in the cubic is the first factor.

For the second and third elements we have from (5) and (6), directly,

$$-\frac{a+b}{2}+\frac{a-b}{2\sqrt{-3}}=\sqrt[3]{-\frac{B}{2}+\sqrt{\frac{A^3}{27}+\frac{B^2}{4}}}, \tag{11}$$

and

$$-\frac{a+b}{2}-\frac{a-b}{2\sqrt{-3}}=\sqrt[3]{-\frac{B}{2}-\sqrt{\frac{A^3}{27}+\frac{B^2}{4}}}. \tag{12}$$

These functions are closely associated with the imaginary cubic roots of unity, viz., $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$ and $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$, but not as factors. In (10) if we substitute any true roots, as a and b, the radical changes sign, but in (11) and (12) the cube root radicals simply interchange places. The cyclic substitutions of the roots all at a time gives in their order the formulas for all the roots the elements remaining unchanged.

In the equations (11) and (12) the reducible case requires that the imaginary term should vanish. The analysis of the term $\frac{a-b}{2\sqrt{-3}}$ with this end in view postulates the conditions which must subsist between the roots in the reducible case, which is obviously that two roots be equal, or that they be conjugate imaginaries, in which case the imaginary factor disappears by division.